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On the phase diagram of spin glasses

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Abstract. We try to explain some common features of the spin-glass phase diagrams found so far using Nishimori's method for $\pm J$ and Gaussian distributions in Ising models on hypercubic lattices.

In recent years there has been much interest in the theory of spin glasses (for a recent review see, for example, Binder and Young (1986)). These models contain random ferromagnetic and antiferromagnetic bonds and the competition between them causes a new type of ordering (Edwards and Anderson 1975). The Ising spin glass is one of the simplest of these models which shares the characteristic spin-glass features.

The model contains ferro- and antiferromagnetic bonds with a given distribution, but it is usually assumed that the main features do not depend strongly on the concrete distribution function. So we will restrict ourselves to the Gaussian and the $\pm J$ distributions in most of this paper. (The spins are thought to be situated on a hypercubic lattice.)

The infinite range spin glass (Sherrington and Kirkpatrick 1975) has been discussed in detail in the literature (see the review of Binder and Young (1986)). We have considerably less information about low-dimensional models and most of that comes from approximate methods. A recent high-temperature series expansion work (Singh and Chakravarty 1986) gives no phase transition in two dimensions but gives a finite critical temperature in higher dimensions for the symmetric $\pm J$ distribution. These findings are in agreement with numerical work (Bhatt and Young 1985, McMillan 1985, Bray and Moore 1984, 1985, Ogielski and Morgenstern 1985).

In this paper we consider phase diagrams of Ising spin glasses. Let us first summarise what is known about these phase diagrams.

The phase diagram of the SK model can be seen in figure 1. The discussion of the diagram can be found in a paper by Toulouse (1980). Besides, we know from Nishimori's work that the boundary of that phase in which the magnetisation is not zero is re-entrant-like (Nishimori 1981).

Moreover, we can get phase diagrams from real space renormalisation group treatments (Jayaprakash *et al* 1977). Although their results are not very reliable, since they obtain a finite critical temperature in two dimensions, it is worth mentioning that the SG–ferro phase boundary is a straight line and the SG–para boundary is also straight around $p=\frac{1}{2}$ ($\pm J$ model) in three and four dimensions. Later we will argue that these two features are generally characteristic of the $\pm J$ model.

In this paper we will use Nishimori's result (Nishimori 1981) and here we sum up his findings. For a given family of distributions (which includes both the $\pm J$ and the Gaussian ones) we can obtain, for a gauge-invariant (Toulouse 1977) quantity A,

$$\overline{\langle A\rangle_{\beta_0}}^{\,p}=\overline{\langle A\rangle_{\beta_0}Z(\beta)/C(\beta)}^{\,1/2} \qquad (1)$$

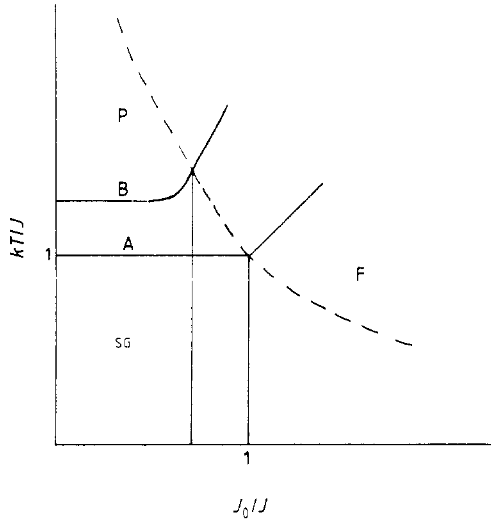

Figure 1. Phase diagram of the SK model for Gaussian distribution (A) and the suggested diagram in d dimensions for a Gaussian distribution (B). The broken curve is the Nishimori line.

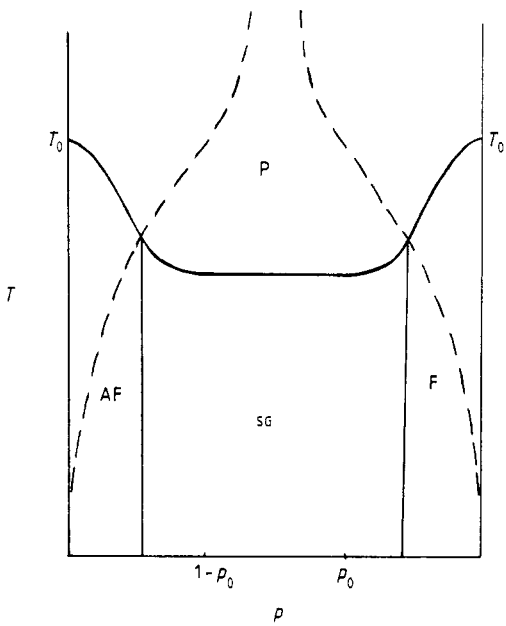

Figure 2. The suggested diagram for $\pm J$ distributions.

where $\langle\ \rangle$ denotes the thermal average, $\overline{}^{p}$ the configurational average with a concentration p of ferromagnetic bonds, β_0 is the inverse temperature of the system and $C(\beta)$ is a constant which is for the $\pm J$ model $C(\beta)=2^{-N}(\cosh\beta J)^{-N_b}$. $Z(\beta)$ is the partition function at the inverse temperature β and $\beta=\beta(p)$ is a function of the concentration (for $\pm J$, $J\beta=\frac{1}{2}\ln(p/1-p)$). We can define a line on the p–T plane which is called the Nishimori line and is characterised by the expression $\beta_0=\beta(p)$. For the magnetisation we can obtain

$$\overline{\langle s_i\rangle_{\beta_0}}^{p}=\overline{\langle s_i\rangle_{\beta_0}\langle\sigma_i\rangle_\beta}^{p} \tag{2}$$

where $\langle\sigma_i\rangle_\beta$ is the magnetisation at the inverse temperature β.

In the following we would like to show that the transition temperature between the disordered (high temperature) and ordered (low temperature) phase is a monotonic function of p on the interval $(0.5, 1)$.

To see this we shall use the replica trick. First let us consider the Gaussian case:

$$\overline{Z^n}=\mathrm{Tr}_{(n)}\exp\left(\beta^2J^2\sum_{\langle ij\rangle}\sum_{\alpha,\beta}s_i^\alpha s_i^\beta s_j^\alpha s_j^\beta+\beta J_0\sum_{\langle ij\rangle}\sum_{\alpha}s_i^\alpha s_j^\alpha\right) \tag{3}$$

where n is the number of replicas, J_0 is the mean, J is the width of the distribution and α,β are replica indices. After averaging, we have an Ising model with two- and four-spin couplings, but all the bonds are non-negative. We therefore use a corollary of the Griffiths inequality (Griffiths 1972) that the critical temperature does not decrease if we strengthen the bonds. Since J_0 is an increasing function of p, we conclude that the transition temperature does not decrease with increasing p. This holds for all n so we assume that it will be valid when we make a continuation in n.

For the $\pm J$ and other models we cannot exponentiate $\overline{Z^n}$ so easily but it can be seen immediately that the effective model (after averaging) is not frustrated and the ground-state energy decreases with increasing p. So the above conclusion is true for all distributions on hypercubic lattices, i.e. the transition temperature is an increasing function of p on the given interval.

Now we would like to see how the phase boundary begins to rise in the $\pm J$ model. Let us investigate the Edwards–Anderson parameter

$$\overline{\langle s_i\rangle^2_{\beta_0}}^{p}=\overline{[Z(\beta)/C(\beta)]\langle s_i\rangle^2_{\beta_0}}^{1/2}. \tag{4}$$

As long as β is low enough we can expand Z/C in powers of $\tanh(\beta J)$. $C(\beta)$ is analytic at all temperatures and $Z(\beta)$ is also analytic for all $\beta<\beta_c$, where $\beta_c=1/kT_c$ and T_c is the critical temperature of the pure ferromagnetic system. (It is possible that $Z(\beta)$ is analytic for $\beta<\beta^*(>\beta_c)$ for almost all bond configurations, but the existence of β^* is not clear.)

$$\frac{Z(\beta)}{C(\beta)}=\sum_{l=0}^{\infty}a_l(\tanh\beta J)^l \tag{5}$$

and

$$\overline{\langle s_i\rangle^2_{\beta_0}}^{p}=\overline{\sum_{l=0}^{\infty}a_l\langle s_i\rangle^2_{\beta_0}(\tanh\beta J)^l}^{1/2}. \tag{6}$$

Since the series is convergent we can write

$$\sum_{l=0}^{\infty}\overline{a_l\langle s_i\rangle^2_{\beta_0}}^{1/2}(\tanh\beta J)^l=\sum_{l=0}^{\infty}b_l(\tanh\beta J)^l. \tag{7}$$

Here the b_l depend only on the $p=\frac{1}{2}$ behaviour. Besides, we know that $2p-1=\tanh\beta J$, so we have

$$\overline{\langle s_i\rangle^2_{\beta_0}}^{\,p}=\sum_{l=0}^{\infty} b_l(2p-1)^l. \tag{8}$$

Let β_0 be such that

$$\overline{\langle s_i\rangle^2_{\beta_0}}^{\,1/2}=0$$

then there exists an $\varepsilon>0$ for which

$$\overline{\langle s_i\rangle^2_{\beta_0}}^{\,p}=0$$

if $p\in(0.5, 0.5+\varepsilon)$. This means that in equation (8) the power series will give 0 on the interval $(0.5, 0.5+\varepsilon)$. To get another value, a singularity must occur in p but the b_l do not depend on p so there is no singularity.

This means that $\overline{\langle s_i\rangle^2_{\beta_0}}^{\,p}=0$ will hold as long as Z/C can be expanded in a power series, i.e. up to the first singularity. As a result the phase boundary must be a straight line around $\frac{1}{2}$ (figure 2).

It is generally believed that in random systems weak singularities can occur (Griffiths 1969) which affect the dynamics of the system. From the above statements we can conclude that these singularities can also be seen in the thermodynamics.

Let us now turn to the characterisation of the ferromagnetic phase. To do so we will make a conjecture for which we give some arguments. We know from Nishimori's work that

$$\overline{\langle s_i\rangle_{\beta_0}}^{\,p}=\overline{\langle s_i\rangle_{\beta_0}\langle\sigma_i\rangle_\beta}^{\,p}. \tag{9}$$

Let B denote the non-gauge-invariant part of the distribution function. Then we can write

$$\overline{\langle s_i\rangle_{\beta_0}}^{\,p}=\overline{\langle s_i\rangle_{\beta_0}B}^{\,1/2}$$

and

$$\overline{\langle\sigma_i\rangle_\beta}^{\,p}=\overline{\langle\sigma_i\rangle_\beta B}^{\,1/2}$$

(for the $\pm J$ model $B=(\cosh\beta J)^{-N_b}\Pi_{\langle ij\rangle}\exp(\beta J_{ij})$). Since $\overline{\langle s_i\rangle_{\tilde\beta}}^{\,1/2}=0$ for all $\tilde\beta$ we can conclude that B and $\langle s_i\rangle_{\beta_0}$ (and also B and $\langle\sigma_i\rangle_\beta$) are correlated, so we can assume that $\langle s_i\rangle_{\beta_0}$ and $\langle\sigma_i\rangle_\beta$ are also correlated so that the two are of the same sign at each site for almost all the configurations, i.e.

$$\overline{\langle s_i\rangle_{\beta_0}}^{\,p}=\overline{|\langle s_i\rangle_{\beta_0}\langle\sigma_i\rangle_\beta|}^{\,p}. \tag{10}$$

Since $\langle s_i\rangle_{\beta_0}$ and $\langle\sigma_i\rangle_\beta$ are positively correlated we have

$$\overline{|\langle s_i\rangle_{\beta_0}\langle\sigma_i\rangle_\beta|}^{\,p}\geqslant\overline{|\langle s_i\rangle_{\beta_0}|}^{\,p}\,\overline{|\langle\sigma_i\rangle_\beta|}^{\,p}. \tag{11}$$

Therefore we conjecture

$$\overline{\langle s_i\rangle_{\beta_0}}^{\,p}\geqslant\overline{|\langle s_i\rangle_{\beta_0}|}^{\,p}\,\overline{|\langle\sigma_i\rangle_\beta|}^{\,p} \tag{12}$$

and we know from Nishimori's paper that

$$\overline{\langle s_i\rangle_{\beta_0}}^{\,p}\leqslant\overline{|\langle\sigma_i\rangle_\beta|}^{\,p}. \tag{13}$$

Let us see how these inequalities affect the boundary of that phase in which the magnetisation is non-zero. If a spin-glass phase exists then the boundary is a straight vertical line, as can easily be seen from (12) and (13).

These findings are in agreement with previous results for the SK model (Toulouse 1980) and real space RG treatments (Jayaprakash *et al* 1977). If the spin-glass phase does not exist then our conjecture (12) does not give new information about the boundary.

We have investigated the phase diagram of spin glasses. We have shown that for the $\pm J$ model, the paramagnetic spin-glass boundary is a straight horizontal line, at least on the interval $(0.5, p^*)$, where $2p^*-1=\tanh(J/kT_c)$ where T_c is the critical temperature of the pure model. This means that Griffiths singularities also affect the thermodynamics of the system. Besides this we have made a conjecture and its consequence means that the boundary of the ferromagnetic phase is a straight vertical line if the spin-glass phase exists.

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